# Universes with a cosmological constant

Domingos Soares
Physics Department
Federal University of Minas Gerais
Belo Horizonte, MG, Brasil

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**Abstract**

I present some relativistic models of the universe that have the cosmological constant ($\Lambda$) in their formulation. Einstein derived the first of them, inaugurating the theoretical strand of the application of the field equations of General Relativity with the cosmological constant. One of the models shown is the Standard Model of Cosmology, which presently enjoys the support of a significant share of the scientific community.

## 1 Introduction

The first cosmological model based in General Relativity Theory (GRT) was put forward in 1917 by the creator of GRT himself. Einstein conceived the universe as a static structure and, to obtain the corresponding relativistic model, introduced a repulsive component in the formulation of the field equations of GRT to avoid the collapse produced by the matter. Such a component appears in the equations as an additional term of the metric field multiplied by a constant, the so-called *cosmological constant* (see [1, eq. 10]). The cosmological constant is commonly represented by the uppercase Greek letter ($\Lambda$) and has physical dimensions of 1/length$^2$.

Besides Einstein's static model, other relativistic models with the cosmological constant were proposed. We will see that these models can be obtained from Friedmann's equation plus the cosmological constant, and by means of the appropriate choices of the *spatial curvature* and of the *matter-energy content* of the universe. The cosmologist Steven Weinberg presents several of those models in his book *Gravitation and Cosmology*, in the chapter entitled *Models with a Cosmological Constant* [2, p. 613]. I will show here some of the models discussed by Weinberg and one that he does not discuss, namely, the modern model of the universe in accelerated expansion, many times called the *Standard Model of Cosmology* (SMC).

The classical Friedmann equation [3] has its field of application considerably expanded with the inclusion of the cosmological constant $\Lambda$. The result is presented in the next section. In section 3 I present some features of three models of universe with a cosmological constant, the Einstein static universe, the de Sitter universe and the SMC. Einstein's and de Sitter's universes will be presented, by didactic considerations, in the general frame of the Friedmann equations, but it is always worth reminding that the Friedmann equations were discovered in the years 1920s, that is, after the proposition of those two models. I finish, in section 4, with some general remarks.

## 2 Friedmann's equations with $\Lambda$

The Friedmann equations are the solutions of the field equations of GRT when these are subject to the strong restrictions of symmetry imposed by the Cosmological Principle (CP). The universe of the CP is homogeneous and isotropic, and has matter-energy density $\rho(t)$, i.e., $\rho$ is only a function of the cosmic time (see section 4 of [1]). Such constraints enormously simplify the field equations. For example, the energy-momentum tensor is reduced to a diagonal tensor [1, eq. 12].

The energy-momentum tensor occupies the right-hand side of the field equations [1, eq. 4]. For the left side we need the space-time metric that describes the physical system in question. In 1935, the American physicist-mathematician Howard Robertson (1894-1975) and, almost simultaneously, the English mathematician Arthur Walker (1909-2001) derived, from purely geometrical arguments, the mathematical expression of the space-time metric of a fluid that obeys the CP. According to Harrison [4, p. 285], they *"showed rigorously that universes obeying the CP have a spacetime that uniquely separates into a curved expanding space and a cosmic time that is common to all comoving observers"* (i.e., that partake the expansion; see also section 3 of [5], where the mathematical expression of the Robertson-Walker metric is presented).

The solutions of the field equations for the Robertson-Walker metric are the Friedmann equations. They describe in full form the balance of matter and energy (eq. 1) and the dynamics (eq. 2) of the cosmic fluid:

$$\frac{\dot{S}^2}{S^2c^2} + \frac{k}{S^2} - \frac{\Lambda}{3} = \frac{8\pi G\rho}{3c^2}, \tag{1}$$

$$\frac{2\ddot{S}}{Sc^2} + \frac{\dot{S}^2}{S^2c^2} + \frac{k}{S^2} - \Lambda = -\frac{8\pi Gp}{c^4}. \tag{2}$$

$S$ is the *scale factor* of the universe, $k$ is the constant of spatial curvature, $\rho$ is the density of matter-energy and $p$ is the pressure of the cosmic fluid (see more details below eq. 12 of [1]). The curvature constant is, for a **closed** and spherical universe, $k = +1/R^2$, where $R$ is the curvature radius of the spherical space. For a **critical** (or **flat**) model, $R \to \infty$, and therefore, $k = 0$. The **open** universe has an imaginary curvature radius, implying in a negative curvature constant, $k = -1/R^2$ (hyperbolic space). Notice that, like the cosmological constant, the curvature constant has physical dimensions of $1/\text{length}^2$.

The Friedmann equations so obtained, from the field equations plus Einstein's cosmological term (also eqs. 16 and 17 of [1]) are the basis of the majority of modern cosmological models, with or without $\Lambda$. For the latter case, it suffices to make $\Lambda = 0$ in eqs. 1 and 2. In the next section I present three models of the universe **with** $\Lambda$.

# 3 Relativistic models with $\Lambda$

I present here three *universes* with $\Lambda$. The first of them is Einstein's static universe, the pioneer model of modern relativistic cosmology. The cosmological constant was introduced by Einstein in his field equations to produce a repulsive energetic component in order to exactly compensate the attractive energetic component originated in the matter-energy content of the universe. Einstein had no reasons, in 1917, to imagine an universe that was not static on large scale. In 1930, the British astrophysicist A. Eddington (1882-1944) conclusively showed that the model was unstable, casting capital doubts about its viability (see the discussion of the energetic stability of Einstein's static model in section 3 of [6]).

The second of them is a dynamical model, but with a strange quality: it does not contain matter, only the energetic component originated with the introduction of the cosmological constant. Despite its strangeness, such a model can be useful for representing the real universe. After all, nothing impedes that the matter content of the universe be, not exactly null but, approximately null, i.e., *mathematically negligible*. And even so, sufficient to create galaxies, stars, planets, humans, etc. The responsible for this model was the Dutch physicist, mathematician and astronomer Willem de Sitter (1872-1934). de Sitter's model was proposed also in 1917, shortly after the proposition of Einstein's universe.

The third model presented is the SMC, the Standard Model of Cosmology, the theoretical universe accepted by considerable parcel of the international scientific community, in spite of its innumerable problems — acknowledged even by those that accept it (some of them are listed in section 3 of [7]). The main feature of this model is having, in the present cosmic time, an accelerated cosmological expansion. Incidentally, note that the models of Friedmann, without $\Lambda$, have decelerated cosmological expansion in all cosmic times (cf. [3]).

## 3.1 Einstein's static universe

Einstein was guided by its physical intuition and by his preconceived ideas about the universe, when he put forward the first relativistic cosmological model. In the 1910s, the universe appears to be a static structure on the largest scale. The ideas of a dynamical universe on large scales, both theoretical and observational, only emerged in the 1920s. Besides static, the universe might certainly be finite. This avoids uncomfortable boundary conditions in the limits of the universe, if it was infinite.

Einstein's universe is therefore finite — it has positive spatial curvature, i.e., it is closed — and static [6].

To obtain such an universe, Einstein was forced to introduce a repulsive component in his field equations. He put an additional term of the metric field, multiplied by the *cosmological constant* $\Lambda$. The value of $\Lambda$ can be fine-tuned to simultaneously get a closed universe (nonzero and positive spatial curvature) and static, that is, $dS/dt \equiv \dot{S} = 0$ (see eqs. 1 and 2 and section 2 of [6]).

The main problem of Einstein's model is its instability. The space is spherical with curvature radius, in the 4-dimensional hyperspace, defined by the matter density of the universe. Now, this situation represents an *unstable* equilibrium; any small perturbation leads the system — the universe — to the collapse or to the disintegrative expansion to infinity. Fig. 1 illustrates that instability by means of a Newtonian analogy. The figure shows the sum of the components of (gravitational) attractive energy and of (cosmological constant) repulsive energy. One clearly sees the instability condition represented by the position of equilibrium at the scale factor $R_E = 1$.

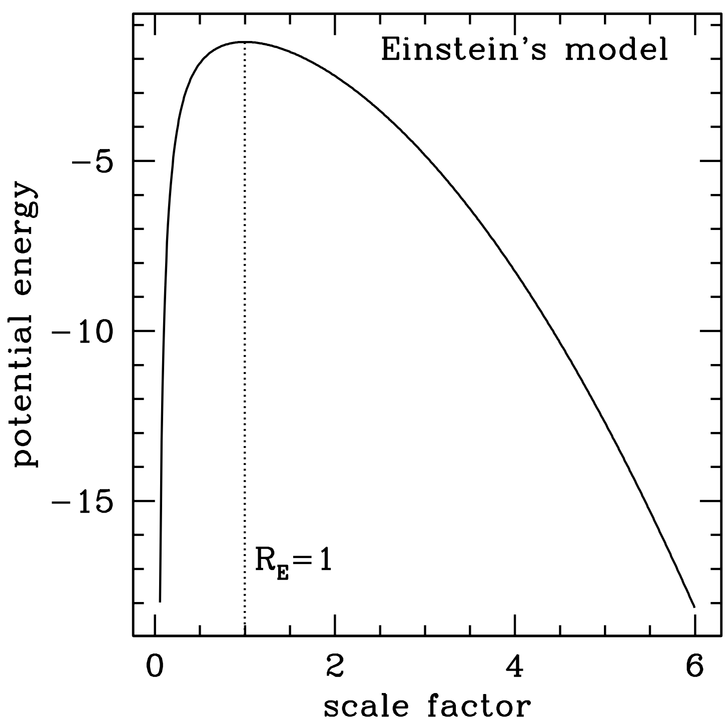


Figure 1: Λ-shaped diagram: the potential energy — in arbitrary units — for the Newtonian analogy of Einstein's static model. Notice that the equilibrium at $R = R_E$ is an unstable one. Any small perturbation at $R_E$ makes either the universe to collapse or diverge to R $\to \infty$ [6].

It is worthwhile emphasizing that Einstein's static model has **nonzero and positive spatial curvature constant**. Other quantitative and qualitative aspects are discussed in more detail in [6].

### 3.2 de Sitter's universe

It is obtained making $k = \rho = 0$ in eq. 1. One gets $\dot{S}(t)/S(t) = c(\Lambda/3)^{1/2}$. The Hubble expansion "constant", or parameter, $H$ is defined as $H \equiv \dot{S}(t)/S(t)$. We see then that de Sitter's model is characterized by a constant relative rate of expansion — the Hubble parameter. This is not true in general. For example, the classical models of Friedmann (cf. [3]) have $\dot{S}(t)/S(t) \equiv H(t)$; the Hubble parameter is *approximately* constant only for small ranges of cosmic time $\Delta$t around any time $t$.

Let the present cosmic time be $t_\circ$ and $S(t_\circ) \equiv 1$. After the integration of the differential equation shown in the previous paragraph, the scale factor in de Sitter's universe can be written as:

$$S(t) = \mathrm{e}^{\mathrm{H}(t-t_\circ)}, \tag{3}$$

with $H = c(\Lambda/3)^{1/2}$. de Sitter's model represents a universe of infinite age, in exponential expansion, with constant Hubble's parameter determined by the cosmological constant $\Lambda$ and the speed of light $c$.

De Sitter faced great resistance to his model, even many years after its proposition. A 1930 Dutch newspaper displayed a cartoon that mocked the Desitterian idea (see Fig. 2).

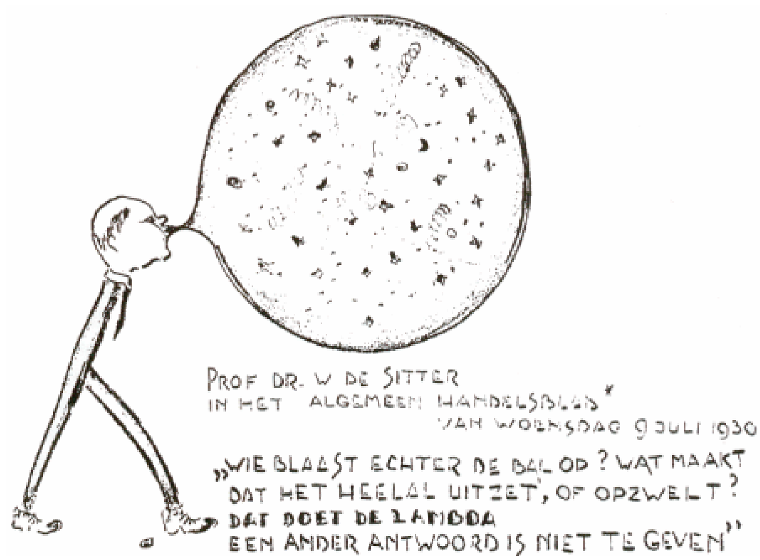


Figure 2: Cartoon in a 1930 Dutch newspaper. Notice that de Sitter's body has the shape of the lowercase Greek letter $\lambda$ (cf. [8]). Translation of the legend: PROF. W. DE SITTER ON THE WEDNESDAY JULY 9, 1930 GENERAL COMMERCE NEWSPAPER "WHO DOES ACTUALLY BLOW UP THE BALLOON? WHAT DOES MAKE THAT THE UNIVERSE EXPANDS, OR SWELLS? IT IS LAMBDA THAT DOES THIS. ONE OTHER ANSWER IS NOT TO GIVE."

Notice that the cartoon is *conceptually wrong* because de Sitter's universe is spatially flat and not curved, like the balloon that appears in the figure. On the other hand, if the curvature radius of the balloon is very large ($\rightarrow \infty$), de Sitter's universe can be imagined as a cap of this hypersphere, because then it will be approximately flat.

De Sitter's universe has an **accelerated** expansion, as can be seen in Fig. 3. This can be be verified in two ways. Qualitatively, the shape of the curve $S(t)$ has the temporal derivative $\dot{S}$, i.e., the tangent to the curve, increasingly larger as time passes. Quantitatively, we know that the upwards concavity of the curve implies that the second derivative of $S(t)$ is **positive**, in other words, $\dot{S}$ increases with time. Note that all classical Friedmann models have **decelerated** expansion, which can be verified by these same reasonings (see [3]).

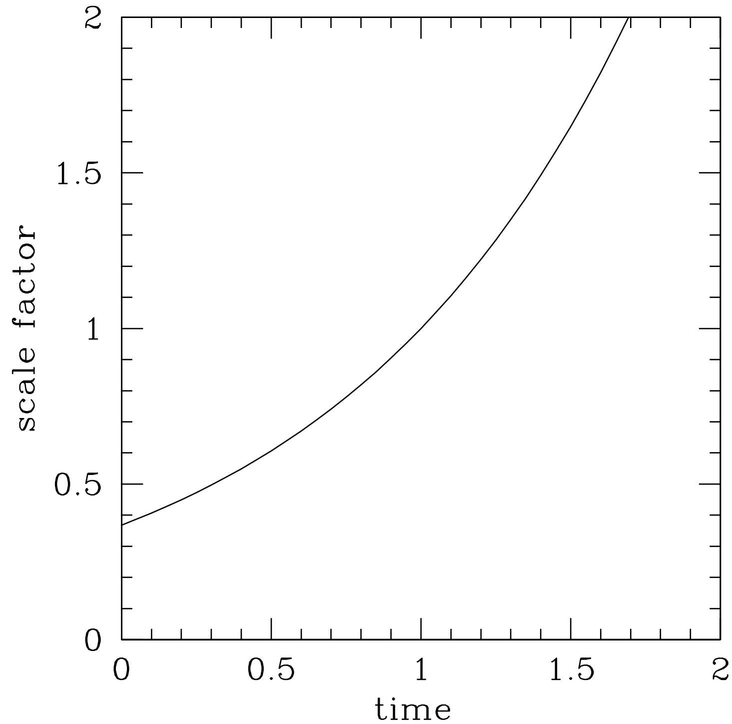


Figure 3: The scale factor for the de Sitter universe increases exponentially with time (cf. eq. 3). The expansion is accelerated in all cosmic times.

De Sitter's model became famous for being the first expanding model. The expansion was called at the time **"de Sitter's effect"**. It was a very strange model, as we saw, due to the fact of not containing matter, i.e., for having **zero matter density**. It is worthwhile point out also that de Sitter's model has **zero spatial curvature constant**.

## 3.3 Standard Model of Cosmology

The SMC represents an universe that has matter and energy associated to the cosmological constant at the exact proportion to yield zero spatial curvature, that is, to be characterized, on large scales, by a **flat spatial geometry**, also called **Euclidean**. This requires that the sum of the densities of matter and of the energy associated to the cosmological constant be exactly equal to the critical density $\rho_c$. Expressed in another way, $\rho_c = \rho_m + \rho_\Lambda$.

With the aid of the *density parameter* $\Omega_m \equiv \rho_m/\rho_c$ and $\Omega_\Lambda \equiv \rho_\Lambda/\rho_c$, one can state that the SMC obeys the relation $\Omega_m + \Omega_\Lambda = 1$. At the present time $t = t_\circ$, it follows that $\Omega_{m\circ} + \Omega_{\Lambda\circ} = 1$, with $\Omega_{m\circ} \equiv \rho_{m\circ}/\rho_{c\circ}$ and $\Omega_{\Lambda\circ} \equiv \rho_{\Lambda_\circ}/\rho_{c\circ}$, being all quantities evaluated at $t = t_\circ$ (more details in [9, cap. 8] and [10, cap. 29]).

Eq. 1 can be written in terms of $\Omega_{m\circ}$ and $\Omega_{\Lambda\circ}$ [10, cap. 29]. The resulting differential equation must be integrated to obtain $S(t)$ for the SMC. The solution of the integral ([9, eq. 8.4][1]) gives the function $S(t)$:

$$t = \left(\frac{2}{3H_\circ}\right)\frac{1}{\Omega_{\Lambda\circ}^{1/2}}\ln\left[\left(1+\frac{\Omega_{\Lambda\circ}}{\Omega_{m\circ}}S(t)^3\right)^{1/2} + \left(\frac{\Omega_{\Lambda\circ}}{\Omega_{m\circ}}S(t)^3\right)^{1/2}\right], \tag{4}$$

where $H_\circ$ is Hubble's constant at $t = t_\circ$, i.e., at the present time. Fig. 4 shows the function $S(t)$ with $\Omega_{m\circ} = 0.3$ and $\Omega_{\Lambda\circ} = 0.7$, figures close to those adopted by the SMC in the current scientific literature.

The age of this universe is $t_\circ = 1.45 \times 2/(3H_\circ)$, as shown in the figure. To get the value of 1.45 just do $S(t = t_\circ) = 1$, $\Omega_{m\circ} = 0.3$ and $\Omega_{\Lambda\circ} = 0.7$ in eq. 4. The quantity $2/(3H_\circ)$ corresponds to the age of the Friedmann model with $\Omega_{m\circ} = 1$ and $\Omega_{\Lambda\circ} = 0$. This model is known as *critical Friedmann model* or *Einstein-de Sitter model* (cf. [3, 11]).

[1] Notice that the numerator of the integrand of eq. 8.4 in Ref. [9] must be corrected to $R^{1/2}dR$.

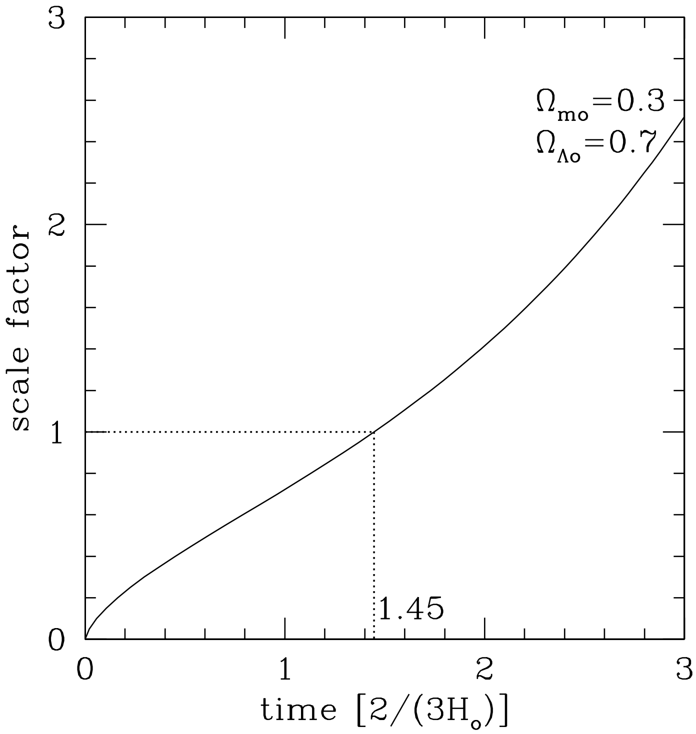


Figure 4: The diagram shows the scale factor for the SMC, which represents an universe with an accelerated phase in recent cosmic times. The age of the universe in this model is shown and corresponds to the unit scale factor. Notice the change of concavity of the curve a little before of the scale factor equal to 1, which indicates that the expansion switched from a decelerated phase to an accelerated one.

The larger the value of $\Omega_{\Lambda\circ}$, the larger the model's age. The SMC does not allowed total freedom in the choice of $\Omega_{\Lambda\circ}$, because there are evidences of the existence of about 30% of matter in the universe ($\Omega_{m\circ} = 0.3$). In any case, the value of $\Omega_{\Lambda\circ} = 0.7$ results in a plausible age for the universe. That is, the cosmological age is approximately equal to the age of the oldest objects in our galaxy (see [11, 12]).

Incidentally, a precursory model of SMC was the semi-qualitative model of the Belgian cosmologist Georges Lemaître (1894-1966), put forward in 1947, and that has three phases: (i) initial phase of decelerated expansion, (ii) static phase or of stagnation and (iii) final phase of accelerated expansion. With the exception of the static phase, this model is a replica of the SMC and allows for the construction of models with ages compatible with the ages of the oldest observed objects. The SMC has, instead of a phase of stagnation, a rapid phase of transition from a decelerated to an accelerated expansion, as we saw in Fig. 4 (see more details in [12]).

Now, it is important to stress that $\Omega_{m\circ} = 0.3$ means that 30% of all the matter-energy content of the universe is in the form of matter, but it is divided between *baryonic matter* (protons and neutrons) and *nonbaryonic matter* (exotic). Indeed, the SMC predicts approximately 5% of baryonic matter and approximately 25% of exotic, nonbaryonic matter. Matter **cannot** be all baryonic, because then the amount of baryonic matter in the time of the synthesis of the light chemical elements would be much larger than the required by the model. From this arises the prediction of the existence of exotic matter (more details in [7, 13]).

Finally, in order to compare with the other models presented, the SMC is a model with **zero spatial curvature constant** and **nonzero matter density**.

# 4 Final remarks

In 1958, Arthur Eddington, in his influential book *The Expanding Universe*, made the famous comparison between Einstein's and de Sitter's model: *"The de Sitter Universe contains motion without matter, while the Einstein Universe contains matter without motion"* [14, p. 46]. In spite of not containing matter in its theoretical formulation, it is possible to imagine, as we saw, *test bodies*, of negligible masses, present in the de Sitter universe. And these bodies have expanding motion because the space itself is in expansion, as mentioned in section 3.2.

The Einstein and the de Sitter models became cornerstones in the construction of the theoretical building of modern cosmology.

In the years 1920s arose the idea of an expanding universe. Einstein's static universe soon became just a theme of the history of cosmology. It appears often in the literature the statement that Einstein would have said, in this context, that the cosmological constant was the greatest of his "blunders". However, as already suggested in Soares [6, sec. 4], Einstein's real great blunder was not the adoption of the cosmological constant, but rather the *formulation of an unstable cosmological model*. The cosmological constant is perfectly acceptable in the expression of the GRT field equation, representing no error whatsoever, from the formal point of view.

The three models with $\Lambda$ presented in section 3 can be characterized by their spatial geometries and by their matter contents.

> Section 3.1) **Einstein's static universe**: nonzero positive *spatial curvature constant*, resulting in a spherical 3-D space. The *matter content* is linked to the cosmological constant with the aim of having a static universe.
>
> Section 3.2) **de Sitter's universe**: zero *spatial curvature constant*, resulting in a flat or Euclidean 3-D space. The *matter content* is zero. The space-time tissue has an accelerated expansion in the interval $-\infty < t < \infty$.
>
> Section 3.3) **Standard Model of Cosmology**: zero *spatial curvature constant*, resulting in a flat or Euclidean 3-D space. The *matter content* is nonzero. The energy associated to the cosmological constant comes to dominate in the present time and forces the universe into a transition to an accelerated expansion.

In 2002, the American cosmologist Michael Turner coined the term **dark energy** to represent the energy associated to the cosmological constant, and that would be responsible for the accelerated expansion of the universe (later studies enlarged the concept of "dark energy" to include other hypothetic forms of energy not associated to the cosmological constant, most of them being not constant). Why **dark**? Because until present times this new kind of energy — totally different from electromagnetic energy, for example — remains completely unknown, both from the observational point of view and from the theoretical characterization standpoint (see [7, 15]).

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