From Schwarzschild to Newton

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Abstract

The only exact solution of General Relativity Theory of recognized practical applicability was obtained by Karl Schwarzschild in 1916. In the limit of weak field it reduces to Newton's gravitational law. I present some aspects of *Schwarzschild's gravitation* and show its reduction to the classical law of gravitation.

1   Introduction

Albert Einstein (1879-1955) completed the formulation of his theory of gravitation, the General Relativity Theory (GRT), in November 1915. The German astrophysicist Karl Schwarzschild (1873-1916) read Einstein's article, published in a Prussian scientific journal, in dramatic circumstances: he was serving the German army during World War I in the Russian front. Immediately he worked out the consequences of the theory for the gravitation caused by an isolated star. He obtained hence the first exact solution of Einstein's field equations for the *exterior space* of a static and spherical distribution of mass M. He sent his results to Einstein who admired Schwarzschild's achievement: he himself did not believe that it was possible to derive an analytical and exact solution of his equations given the enormous mathematical complexities involved. Schwarzschild had the ingenuity of choosing a simple and highly symmetrical system in order to derive the first and most notable particular solution of GRT [1, p. 124].

Schwarzschild's work was published in January 1916. A few weeks later he found the solution for the *interior* of the mass M. In May of that same year, as a consequence of an illness he acquired in the war, Schwarzschild's brilliant scientific career was brought to an end by his premature death.

According to Rindler [2, p. 228], Schwarzschild's external solution remains, until today, as the most important exact solution of Einstein's field equations, given its vast applicability to real systems. The most known applications are the calculation of planetary orbits in strong gravitational fields, being the correct

prediction of the annual advance of Mercury's perihelion the most known example, the calculation of the bending of light's trajectory while passing near very massive bodies, the calculation of the gravitational redshift, etc. One might not forget to mention also the opening up of a large and new field of research, with the consideration of the objects called *black holes*. These appear as limiting structures of Schwarzschild's metric. The black hole remains, nonetheless, as a "working hypothesis", which awaits, for its thorough and precise discussion, the creation of a theory of quantum gravity.

Schwarzschild's solution of the general equations of Einstein's gravitation for the exterior space of a spherical and static mass M will be henceforth called *Schwarzschild's gravitation* in a counterpoint to *Newton's gravitation*, which is the limit of Schwarzschild's gravitation for large distances from the mass M (the so-called *region of weak field*).

GRT's high prestige comes fundamentally from Schwarzschild's gravitation, which sometimes is, in a mistaken way, confused with GRT itself, that is to say, the part is taken by the whole.

In the next section, I present Schwarzschild's metric and show that it becomes Minkowski's metric, i.e., the flat space-time metric of Einstein's Special Relativity Theory (SRT), for infinitely large distances from the mass M. In the third section, I show how Schwarzschild's gravitation reduces to Newton's classical gravitation when distances from the mass M are large, but not *infinitely large*. I present in section 4 some final remarks.

2 Schwarzschild's metric

The *metric field* or simply the *metric* of any space of N dimensions tells us quantitatively how to measure distances in this space. The spaces of GRT and SRT have N=4 dimensions. They are, in Cartesian coordinates, t, x, y and z, one time coordinate and 3 spatial ones. Such a four-dimensional space is called *space-time*.

SRT's space-time is flat (also called Euclidean) and its metric is called *Minkowski's metric* (see [3]). Minkowski's metric, in Cartesian coordinates, characterizing it as a flat space, is given by the infinitesimal distance between two *events* located at (t, x, y, z) and $(t + dt, x + dx, y + dy, z + dz)$:

$$(ds)^2 = -(cdt)^2 + (dx)^2 + (dy)^2 + (dz)^2, \qquad (1)$$

where c is the speed of light in vacuum. In spherical coordinates r, θ and ϕ it is written as:

$$(ds)^2 = -(cdt)^2 + (dr)^2 + (rd\theta)^2 + (r\sin\theta d\phi)^2. \qquad (2)$$

The metrics shown in Eqs. 1 and 2 can also be written in matrix format and, in this case, one has the so-called *metric tensors* $g_{\mu\nu}$. They are, respectively:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3}$$

and

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & (r\sin\theta)^2 \end{pmatrix}. \tag{4}$$

The indices μ and ν take the values 0, 1, 2 and 3 corresponding to the time coordinate and to the three spatial coordinates. As can be seen in Eqs. 3 and 4, Minkowski's metric tensor has only the diagonal terms, which means that time and space are *isotropic* (coordinate axes are identical in any direction) and *homogeneous* (the axes are identical in any point of space-time). This is also true for Schwarzschild's metric, as I show below.

SRT is a geometrical theory of flat space-time (Minkowski's metric). GRT's metrics characterize curved space-times. Schwarzschild's metric is the metric corresponding to the exterior space of a spherical distribution of mass, without rotation. The Sun, and many other stars and planets, can be approximated by such a distribution. Next, I show how its mathematical expression is obtained by means of GRT.

GRT's prime equation is Einstein's field equation, which is a tensor equation that relate 4×4 tensors (the *field* here is the metric field $g_{\mu\nu}$; see more details in [4]). The full Einstein's field equations, therefore, amount to 16 (=4×4) equations and are written in a concise form as [4, Eq. 4]:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{8\pi G}{c^4}T_{\mu\nu}, \tag{5}$$

where $R_{\mu\nu}$ is Ricci's tensor, $g_{\mu\nu}$ is the metric tensor, R is the scalar curvature and $T_{\mu\nu}$ is the energy-momentum tensor; all of these quantities are related to the system to which the equation is applied. G is the universal gravitational constant.

In the case of Schwarzschild's solution $T_{\mu\nu} = 0$, because one is interested in the external region of the mass distribution. Einstein's equation reduces to the vacuum field equations $R_{\mu\nu} = 0$ [4, sec. 3.2]. Rindler [2, pp. 228-229] calculates the diagonal terms of $g_{\mu\nu}$ for the exterior of the static spherical distribution, submitted to $R_{\mu\nu} = 0$. The result is *Schwarzschild's metric*. The metric and the metric tensor are given below in spherical coordinates, which are the most appropriate due to the symmetry of the corresponding physical system, namely, a spherical distribution of mass. The metric is:

$$(ds)^2 = -(1 - 2GM/rc^2)(cdt)^2 + \left(\frac{1}{1 - 2GM/rc^2}\right)(dr)^2 + (rd\theta)^2 + (r\text{sin}\theta d\phi)^2$$

(6)

and the metric tensor is:

$$g_{\mu\nu} = \begin{pmatrix} -(1 - 2GM/rc^2) & 0 & 0 & 0 \\ 0 & \frac{1}{1 - 2GM/rc^2} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & (r\text{sin}\theta)^2 \end{pmatrix}.$$

(7)

Eqs. 6 and 7 represent the metric of a curved space-time, which for distances tending to infinity ($r \to \infty$) reduces to the Minkowski flat space-time of SRT, whose metric is given by Eqs. 2 and 4, as can be easily verified.

3 Schwarzschild's gravitation and Newton's gravitation

The Austrian physicist and cosmologist Wolfgang Rindler comments on the relation between Einstein's gravitation and Newton's gravitation (cf. [5, p. 4-9]). I quote his words below, originally published in the *American Journal of Physics*.

> "Ever since Francis Bacon, it had been believed that the laws of Nature were there to be 'discovered', if only one made the right experiments. Einstein taught us differently. He stressed the vital role of human inventiveness in the process. Newton 'invented' the force of gravity to explain the motion of the planets. Einstein 'invented' curved spacetime and the geodesic law; in his theory there is no force of gravity. If two such utterly different mathematical models can (almost) both describe the same observations, surely it must be admitted that physical theories do not tell us what nature is, only what it is like. The marvel is that nature seems to go along with some of the 'simplest' models that can be constructed ... "

Schwarzschild's gravitation (particular solution of Einstein's GRT) improves Newton's gravitation and in many physical situations they furnish *almost* the same theoretical predictions for the same phenomena. I show in this section the reason of the word "almost" that appears between parentheses in Rindler's text. In other words, I show how Schwarzschild's gravitation — curved-space-time gravitation — reduces to Newton's gravitation — force-of-gravity theory.

In the previous section I mentioned that for $r \to \infty$ (that is, too far away from the mass M) Schwarzschild's metric (Eq. 7) reduces to the flat space-time metric (Eq. 4), which also represents a region where there are no masses, or

they are negligible, in such a way that gravitation is nil. For Einstein, this is a situation in which space-time is flat and for Newton, in which the force of gravity does not exist. In this case both interpretations are equivalent.

Now, before that r "tends to infinity" one has the situation in which r "is large" but not "too large" as $r \to \infty$. One asks: how is Schwarzschild's metric expressed in such a case? I show in what follows that there is a situation in which space-time is just *slightly* curved, which corresponds precisely to Newtonian gravitation. In order to show that, I follow cosmologist Wolfgang Rindler's again (an alternative demonstration is in [6, Lecture 14]).

Rindler [2, p. 188] shows that Schwarzschild's metric can be expressed in terms of the gravitational potential Φ, the language of Newton. The metric given by Eq. 6 becomes:

$$(ds)^2 = -e^{2\Phi/c^2}(cdt)^2 + (dl)^2,\tag{8}$$

in which dl represents the whole spatial part of the metric (notice that for $\Phi = $ constant $= 0$ one recovers Minkowski's metric, Eqs. 1 and 2). The relativistic gravitational potential Φ is obtained by the comparison between Eqs. 6 and 8. One gets [2, p. 230]:

$$\Phi = \frac{c^2}{2} \ln \left(1 - \frac{2}{c^2} \frac{GM}{r} \right).\tag{9}$$

Eq. 9 represents the exact expression of the relativistic gravitational potential equivalent to Schwarzschild's curved space-time. For large distances from the mass M, the term $2GM/rc^2$ is very small and the logarithmic function may be expanded in a power series. The logarithmic expansion is given by:

$$\ln(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots, \; for \; |x| < 1.\tag{10}$$

Making $x \equiv -2GM/rc^2$, the expansion for Φ is obtained:

$$\Phi = -\frac{GM}{r} - \frac{1}{c^2} \left(\frac{GM}{r} \right)^2 - \frac{4}{3} \frac{1}{c^4} \left(\frac{GM}{r} \right)^3 - \ldots\tag{11}$$

Taking only the first term of the expansion one has the Newtonian gravitational potential $\Phi = -GM/r$. In other words, Newtonian gravitation corresponds to a *slightly* curved space-time, as previously stated, because with such an approximation the potential Φ that appears in the metric shown in Eq. 8 is not the whole potential. It must be pointed out that the logarithmic expansion given by Eq. 11 is only valid for the weak gravitational field; in the neighborhood of the mass M, where the field is strong, the exact equation for Φ (Eq. 9) must be used.

Schwarzschild's metric (Eq. 7) may also be reduced to a form corresponding to the Newtonian gravitation. For this it is necessary to make another power series expansion, now using the binomial expansion:

$$(1+x)^n = 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \dots \qquad (12)$$

The term $(1 - 2GM/rc^2)^{-1}$ of Eq. 7 can be expanded using the above power series. Keeping only the first two terms of the expansion one gets $(1 - 2GM/rc^2)^{-1} \cong 1 + 2GM/rc^2$ and Schwarzschild's metric becomes

$$g_{\mu\nu} = \begin{pmatrix} -1 + 2GM/rc^2 & 0 & 0 & 0 \\ 0 & 1 + 2GM/rc^2 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & (r\mathrm{sin}\theta)^2 \end{pmatrix}, \qquad (13)$$

which is the mentioned *slightly*-curved metric, corresponding to the Newtonian gravitation. Notice that it is Minkowski's metric (Eq. 4) with a perturbation $2GM/rc^2$. It is interesting to note also that if one considers one more term of the expansion (Eq. 12), the metric will be "almost" Newtonian, the "almost" that Rindler mentions in the quotation at the beginning of this section. As more terms are considered, the metric increasingly approaches Schwarzschild's metric, and one is closer to the mass M responsible for the curving of space-time (and of space, cf. [3, section 3.1]). The biggest difference between Newton's gravitation and Schwarzschild's gravitation is seen in the regions of strong gravitational field, that is to say, in the near proximity of the mass M; "gravitational field" here has two meanings: to Newton it represents *the gravitational force* and to Schwarzschild it represents *the space-time metric*.

4   Additional remarks

As seen in the previous section, Newtonian gravitation can also be expressed in terms of a curved space-time. This can be done in two ways: (a) it corresponds to Minkowski's space-time *slightly* modified to be non flat, and (b) it corresponds to Schwarzschild's space-time modified to be just *slightly* curved, which occurs far away from the gravitational source (the mass M). Both visions are perfectly described by Eq. 13, which for (a) is Eq. 4 slightly modified, and for (b) is Eq. 7 in the limit of very large r, as shown in the end of section 3.

Schwarzschild's solution — Schwarzschild's gravitation — is the only GRT solution that was experimentally proved in its whole. *Its stunning success is very often taken as the presumed success of all possible GRT's solutions.* One takes the success of the part as the success of the whole. And this is not necessarily true. For example, GRT's cosmological solutions, applied to the cosmic fluid (see [7], [8]), including the current cosmological model, are until now, without exception, non-experimentally proved solutions. The modern cosmological model only survives at the cost of important hypotheses not yet verified, namely,

the hypotheses of the existences of *non-baryonic dark matter*, of *baryonic dark matter* and of *dark energy* (further details in [9]).

GRT is in fact still an uncompleted theory and therefore with limited application. A more complete theory of gravitation has to implement quantum mechanical precepts in its structure, in other words, it must be a theory of *quantum gravitation*. It could have something from GRT in its formulation or it could be a completely different theory from GRT, just like GRT is a completely different theory from the Newtonian theory of gravitation.

This kind of advancement in science is by no means new. It has occurred, for example, with the *electromagnetic theory* of the Scottish physicist and mathematician James Clerk Maxwell (1831-1879). The success of Maxwell's electromagnetism is also spectacular, it has many successful applications, amongst them, the prediction and experimental confirmation that light is an electromagnetic wave. But electromagnetism also found its limits, which were only overcome with the advent of quantum mechanics, as were the cases of the photoelectric effect and the blackbody radiation.

In 1948, the American physicist Richard Phillips Feynman (1918-1988) and others formulated the quantum theory of electricity and magnetism, the *quantum electrodynamics* which, with great success, incorporated quantum mechanics to the classical electromagnetism. Those interested find an extraordinary description of quantum electrodynamics by Feynman himself in the booklet *QED, the strange theory of light and matter* (QED is the acronym for "Quantum ElectroDynamics" [10]).

Einstein himself recognized GRT's limitations, when applied to physical systems that show solutions with singularities. The obvious examples are in the most popular relativistic models of modern cosmology, which exhibit a singularity. Quantum effects are more evident in extreme situations of this sort. Singularities are theoretical aberrations, which disappear when a more complete theory is considered.

References

[1] K. Thorne, *Black Holes and Time Warps: Einstein's Outrageous Legacy*, (W.W. Norton & Company, New York, 1994).

[2] W. Rindler, *Relativity – Special, General, and Cosmological* (Oxford University Press, New York, 2006).

[3] D. Soares, *Espaço e espaço-tempo nas teorias relativistas* (2013), http://www.fisica.ufmg.br/~dsoares/esptmp/esptmp.htm (in Portuguese)

[4] D. Soares, *Physico-mathematical foundations of relativistic cosmology* (2013), arXiv:1309.2252 [physics.gen-ph]

[5] E.F. Taylor, J.A. Wheeler, *Exploring Black Holes: Introduction to General Relativity*, (Addison Wesley Longman, San Francisco, 2000).

[6] S. Waner, *Introduction to Differential Geometry and General Relativity (Lecture Notes)* (2002),
`http://www.zweigmedia.com/diff_geom/tc.html`

[7] A. Viglioni e D. Soares, *Note on the classical solutions of Friedmann's equation* (2011), arXiv:1007.0598v6 [physics.gen-ph]

[8] D. Soares, *Einstein's static universe* (2012), arXiv:1203.4513v2 [physics.gen-ph]

[9] D. Soares, *A stone of stumbling to General Relativity Theory* (2009),
`http://www.fisica.ufmg.br/~dsoares/ensino/grt-stn.pdf`

[10] R. Feynman, *QED: The Strange Theory of Light and Matter* (Princeton University Press, Princeton, 2006).