F

Permutation Group Character Tables

In this appendix we provide tables to be used with permutation groups. Tables F.1 and F.2 are the extended character tables for the permutation groups of 3 and 4 objects $P(3)$ and $P(4)$, respectively, and are discussed in Sects. 17.4.2 and 17.4.3, respectively. The discussion in these sections can also be used to understand the extended character tables for the permutation groups $P(5)$, $P(6)$, and $P(7)$ which have many more symmetry elements, namely $5! = 120$, $6! = 720$, and $7! = 5,040$, respectively (see Tables F.3 and F.4). These character tables are sufficient to describe the permutation groups arising for the filling of s, p, d, and f electron states, as discussed in Chap. 17. In Table F.5 for the group $P(7)$ only a few entries are made. The corresponding entries can also be made for permutation groups $P(n)$ of higher order.

When one considers a wave function of n identical particles (e.g., permutation groups in Chap. 17) then the interchange of identical particles is a symmetry operation that must be included. The number of irreducible representations is equal to the number of classes. Table F.6 contains the number of classes and the dimensionalities of the irreducible representations where $P(n)$ labels the permutation group of n objects.

Table F.1. Extended character table for permutation group $P(3)$

	χ(E)	χ(A,B,C)	χ(D,F)	
$P(3)$	(1^3)	$3(2,1)$	$2(3)$	
Γ_1^s	1	1	1	
Γ_1^a	1	–1	1	
Γ_2	2	0	–1	
$\chi_{\text{perm.}}(\psi_1\psi_1\psi_1)$	1	1	1	$\Rightarrow \Gamma_1^s$
$\chi_{\text{perm.}}(\psi_1\psi_1\psi_2)$	3	1	0	$\Rightarrow \Gamma_1^s + \Gamma_2$
$\chi_{\text{perm.}}(\psi_1\psi_2\psi_3)$	6	0	0	$\Rightarrow \Gamma_1^s + \Gamma_1^a + 2\Gamma_2$

Table F.2. Extended character table for the permutation group $P(4)$

$P(4)$	(1^4)	$8(3,1)$	$3(2^2)$	$6(2,1^2)$	$6(4)$	
Γ_1^s	1	1	1	1	1	
Γ_1^a	1	1	1	−1	−1	
Γ_2	2	−1	2	0	0	
Γ_3	3	0	−1	1	−1	
$\Gamma_{3'}$	3	0	−1	−1	1	
$\chi_{\text{perm.}}(\psi_1\psi_1\psi_1\psi_1)$	1	1	1	1	1	$\Rightarrow \Gamma_1^s$
$\chi_{\text{perm.}}(\psi_1\psi_1\psi_1\psi_2)$	4	1	0	2	0	$\Rightarrow \Gamma_1^s + \Gamma_3$
$\chi_{\text{perm.}}(\psi_1\psi_1\psi_2\psi_2)$	6	0	2	2	0	$\Rightarrow \Gamma_1^s + \Gamma_2 + \Gamma_3$
$\chi_{\text{perm.}}(\psi_1\psi_1\psi_2\psi_3)$	12	0	0	2	0	$\Rightarrow \Gamma_1^s + \Gamma_2 + 2\Gamma_3 + \Gamma_{3'}$
$\chi_{\text{perm.}}(\psi_1\psi_2\psi_3\psi_4)$	24	0	0	0	0	$\Rightarrow \Gamma_1^s + \Gamma_1^a + 2\Gamma_2 + 3\Gamma_3 + 3\Gamma_{3'}$

Here the Γ_{n-1} irreducible representation is Γ_3 (see Sect. 17.3)

Table F.3. Extended character table for permutation group $P(5)$

$P(5)$or S_5	(1^5)	$10(2,1^3)$	$15(2^2,1)$	$20(3,1^2)$	$20(3,2)$	$30(4,1)$	$24(5)$
Γ_1^s	1	1	1	1	1	1	1
Γ_1^a	1	–1	1	1	–1	–1	1
Γ_4	4	2	0	1	–1	0	–1
$\Gamma_{4'}$	4	–2	0	1	1	0	–1
Γ_5	5	1	1	–1	1	–1	0
$\Gamma_{5'}$	5	–1	1	–1	–1	1	0
Γ_6	6	0	–2	0	0	0	1
$\chi_{\text{perm.}}(\psi_1\psi_1\psi_1\psi_1\psi_1)$	1	1	1	1	1	1	1
$\chi_{\text{perm.}}(\psi_1\psi_1\psi_1\psi_1\psi_2)$	5	3	1	2	0	1	0
$\chi_{\text{perm.}}(\psi_1\psi_1\psi_1\psi_2\psi_2)$	10	4	2	1	1	0	0
$\chi_{\text{perm.}}(\psi_1\psi_1\psi_1\psi_2\psi_3)$	20	6	0	2	0	0	0
$\chi_{\text{perm.}}(\psi_1\psi_1\psi_2\psi_2\psi_3)$	30	6	2	0	0	0	0
$\chi_{\text{perm.}}(\psi_1\psi_1\psi_2\psi_3\psi_4)$	60	6	0	0	0	0	0
$\chi_{\text{perm.}}(\psi_1\psi_2\psi_3\psi_4\psi_5)$	120	0	0	0	0	0	0

S_5	irreducible representations
$\chi_{\text{perm.}}(\psi_1\psi_1\psi_1\psi_1\psi_1)$	$\Rightarrow \Gamma_1^s$
$\chi_{\text{perm.}}(\psi_1\psi_1\psi_1\psi_1\psi_2)$	$\Rightarrow \Gamma_1^s + \Gamma_4$
$\chi_{\text{perm.}}(\psi_1\psi_1\psi_1\psi_2\psi_2)$	$\Rightarrow \Gamma_1^s + \Gamma_4 + \Gamma_5$
$\chi_{\text{perm.}}(\psi_1\psi_1\psi_1\psi_2\psi_3)$	$\Rightarrow \Gamma_1^s + 2\Gamma_4 + \Gamma_5 + \Gamma_6$
$\chi_{\text{perm.}}(\psi_1\psi_1\psi_2\psi_2\psi_3)$	$\Rightarrow \Gamma_1^s + 2\Gamma_4 + 2\Gamma_5 + \Gamma_{5'} + \Gamma_6$
$\chi_{\text{perm.}}(\psi_1\psi_1\psi_2\psi_3\psi_4)$	$\Rightarrow \Gamma_1^s + 3\Gamma_4 + \Gamma_{4'} + 3\Gamma_5 + 2\Gamma_{5'} + 3\Gamma_6$
$\chi_{\text{perm.}}(\psi_1\psi_2\psi_3\psi_4\psi_5)$	$\Rightarrow \Gamma_1^s + \Gamma_1^a + 4\Gamma_4 + 4\Gamma_{4'} + 5\Gamma_5 + 5\Gamma_{5'} + 6\Gamma_6$

Here the Γ_{n-1} irreducible representation of $P(5)$ is Γ_4”

Table F.4. Extended character table for permutation group $P(6)$

$P(6)$	1 (1^6)	15 $(2,1^4)$	45 $(2^2,1^2)$	15 (2^3)	40 $(3,1^3)$	120 $(3,2,1)$	40 (3^2)	90 $(4,1^2)$	90 $(4,2)$	144 $(5,1)$	120 (6)
Γ_1^s	1	1	1	1	1	1	1	1	1	1	1
Γ_1^a	1	–1	1	–1	1	–1	1	–1	1	1	–1
Γ_5	5	3	1	–1	2	0	–1	1	–1	0	–1
$\Gamma_{5'}$	5	–3	1	1	2	0	–1	–1	–1	0	1
$\Gamma_{5''}$	5	1	1	–3	–1	1	2	–1	–1	0	0
$\Gamma_{5'''}$	5	–1	1	3	–1	–1	2	1	–1	0	0
Γ_9	9	3	1	3	0	0	0	–1	1	–1	0
$\Gamma_{9'}$	9	–3	1	–3	0	0	0	1	1	–1	0
Γ_{10}	10	2	–2	–2	1	–1	1	0	0	0	1
$\Gamma_{10'}$	10	–2	–2	2	1	1	1	0	0	0	–1
Γ_{16}	16	0	0	0	–2	0	–2	0	0	1	0
$\chi_{\text{perm.}}(\psi_1\psi_1\psi_1\psi_1\psi_1\psi_1)$	1	1	1	1	1	1	1	1	1	1	1
$\chi_{\text{perm.}}(\psi_1\psi_1\psi_1\psi_1\psi_1\psi_2)$	6	4	2	0	3	1	0	1	0	1	0
$\vdots$	$\dots$										
$\chi_{\text{perm.}}(\psi_1\psi_2\psi_3\psi_4\psi_5\psi_6)$	720	0	0	0	0	0	0	0	0	0	0

S_6	irreducible representations
$\Gamma_{\text{perm.}}(\psi_1\psi_1\psi_1\psi_1\psi_1\psi_1)$	$\Rightarrow \Gamma_1^s$
$\Gamma_{\text{perm.}}(\psi_1\psi_1\psi_1\psi_1\psi_1\psi_2)$	$\Rightarrow \Gamma_1^s + \Gamma_5$
$\vdots$	$\vdots$
$\Gamma_{\text{perm.}}(\psi_1\psi_2\psi_3\psi_4\psi_5\psi_6)$	$\Rightarrow \Gamma_1^s + \Gamma_1^a + 5\Gamma_5 + 5\Gamma_{5'} + 5\Gamma_{5''} + 5\Gamma_{5'''} +$ $9\Gamma_9 + 9\Gamma_{9'} + 10\Gamma_{10} + 10\Gamma_{10'} + 16\Gamma_{16}$

Here the Γ_{n-1} irreducible representation of $P(6)$ is Γ_{5e}

Table F.5. Character table (schematic) for group $P(7)$

$P(7)$ or S_7	(1^7)	$\cdots$
Γ_1^s	1	$\cdots$
Γ_1^a	1	$\cdots$
Γ_6	6	$\cdots$
$\Gamma_{6'}$	6	$\cdots$
Γ_{14}	14	$\cdots$
$\Gamma_{14'}$	14	$\cdots$
$\Gamma_{14''}$	14	$\cdots$
$\Gamma_{14'''}$	14	$\cdots$
Γ_{15}	15	$\cdots$
$\Gamma_{15'}$	15	$\cdots$
Γ_{21}	21	$\cdots$
$\Gamma_{21'}$	21	$\cdots$
Γ_{35}	35	$\cdots$
$\Gamma_{35'}$	35	$\cdots$
Γ_{20}	20	$\cdots$
$\chi_{\text{perm.}}(\psi_1\psi_1\psi_1\psi_1\psi_1\psi_1\psi_1)$	1	$\cdots$
$\chi_{\text{perm.}}(\psi_1\psi_1\psi_1\psi_1\psi_1\psi_1\psi_2)$	7	$\cdots$
$\vdots$	$\vdots$	$\vdots$
$\chi_{\text{perm.}}(\psi_1\psi_2\psi_3\psi_4\psi_5\psi_6\psi_7)$	5,040	$\cdots$

S_7	irreducible representations
$\Gamma_{\text{perm.}}(\psi_1\psi_1\psi_1\psi_1\psi_1\psi_1\psi_1)$	$\Rightarrow \Gamma_1^s$
$\Gamma_{\text{perm.}}(\psi_1\psi_1\psi_1\psi_1\psi_1\psi_1\psi_2)$	$\Rightarrow \Gamma_1^s + \Gamma_6$
$\vdots$	$\vdots$
$\Gamma_{\text{perm.}}(\psi_1\psi_2\psi_3\psi_4\psi_5\psi_6\psi_7)$	$\Rightarrow \Gamma_1^s + \Gamma_1^a + 6\Gamma_6 + 6\Gamma_{6'} + 14\Gamma_{14}$ $+14\Gamma_{14'} + 14\Gamma_{14''} + 14\Gamma_{14'''} + 15\Gamma_{15} + 15\Gamma_{15'}$ $+21\Gamma_{21} + 21\Gamma_{21'} + 35\Gamma_{35} + 35\Gamma_{35'} + 20\Gamma_{20}$

Table F.6. Number of classes and the dimensionalities of the Γ_i in $P(n)$

group	classes	number of group elements $\sum_i \ell_i^2$
$P(1)$	1	$1! = 1^2 = 1$
$P(2)$	2	$2! = 1^2 + 1^2 = 2$
$P(3)$	3	$3! = 1^2 + 1^2 + 2^2 = 6$
$P(4)$	5	$4! = 1^2 + 1^2 + 2^2 + 3^2 + 3^2 = 24$
$P(5)$	7	$5! = 1^2 + 1^2 + 4^2 + 4^2 + 5^2 + 5^2 + 6^2 = 120$
$P(6)$	11	$6! = 1^2 + 1^2 + 5^2 + 5^2 + 5^2 + 5^2 + 9^2 + 9^2 + 10^2 + 10^2 + 16^2 = 720$
$P(7)$	15	$7! = 1^2 + 1^2 + 6^2 + 6^2 + 14^2 + 14^2 + 14^2 + 14^2 + 15^2 + 15^2 + 21^2 + 21^2 +$
		$+35^2 + 35^2 + 20^2 = 5,040$
$P(8)$	22	$8! = 1^2 + 1^2 + 7^2 + 7^2 + 14^2 + 14^2 + 20^2 + 20^2 + 21^2 + 21^2 + 28^2 + 28^2 +$
		$+35^2 + 35^2 + 56^2 + 56^2 + 64^2 + 64^2 + 70^2 + 70^2 + 42^2 + 90^2 = 40{,}320$
$P(9)$	31	$9! = 1^2 + 1^2 + 8^2 + 8^2 + \ldots = 362{,}880$
$P(10)$	37	$10! = 1^2 + 1^2 + 9^2 + 9^2 + \ldots = 3{,}628{,}800$
$P(11)$	52	$11! = 1^2 + 1^2 + 10^2 + 10^2 + \ldots = 39{,}916{,}800$
$P(12)$	67	$12! = 1^2 + 1^2 + 11^2 + 11^2 + \ldots = 479{,}001{,}600$
$\vdots$	$\vdots$	$\vdots$
$P(n)$		$n! = 1^2 + 1^2 + (n-1)^2 + (n-1)^2 + \ldots = n!$